J. Fluid Mech. (1969), *vol.* 38, *part* 3, *pp.* 537–546
Printed in Great Britain

The drag on a cloud of spherical particles in low Reynolds number flow

By CHRISTOPHER K. W. TAM
Department of Aeronautics and Astronautics
Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received 22 July 1968 and in revised form 8 April 1969)

A formula for the drag exerted on a cloud of spherical particles of a given particle size distribution in low Reynolds number flow is derived. It is found that the drag experienced by a particle depends only on the first three moments of the distribution function. A treatment of viscous interaction between N particles to the lowest order is carried out systematically. By appealing to the concept of 'randomness' of the particle cloud, equations describing the averaged properties of the fluid motion are derived. The averages are formed over a statistical ensemble of particle configurations. These mean flow equations so obtained are in a form resembling a generalized version of Darcy's empirical equation for the motion of fluid in a porous medium. The physical meaning of these equations is discussed.

1. Introduction

The problem of determining the viscous drag force exerted on a cloud of spherical particles in low Reynolds number flow has been the subject of many theoretical and experimental works for many years (see Happel & Brenner 1965 for references). As yet there does not seem to be available any satisfactory theoretical result which gives the drag force in a particle cloud of a prescribed spectrum of particle sizes. In the special case when all the spherical particles of the cloud have the same radius a widely quoted theory was given by Brinkman (1947). However, in his original treatment Brinkman employed the model of a spherical particle embedded in a porous medium. He described the flow through this porous mass by a modification of Darcy's equation. Since the latter equation is empirical, Brinkman's result has not been generally regarded as a completely rigorous theoretical solution to the problem even though by comparing with experimental data the success of Brinkman's formula is indisputable.

The aim of this paper is to consider slow viscous flow past a large collection of spherical particles of a given size distribution and to derive a particle drag formula free from empirical assumptions. In our present treatment we base our formulation on the Stokes equations, a 'point-force' approximation which we will explain in the appendix, and a certain statistical concept which has been widely used in multiple wave scattering problems (see Foldy 1945; Keller 1964). The essence of the point force approximation is to replace the disturbance produced by a sphere in low Reynolds number flow by that of a point force located at the centre of the sphere. The force is taken to be equal in magnitude but opposite in direction to the drag on the particle. By means of this approximation we will first derive a set of self-consistent equations governing the viscous interaction between a large number of particles. On taking the ensemble average of this set of equations we obtain averaged equations describing the mean fluid flow with particles embedded in it. The mean flow equations turn out to be in the form of a generalized Darcy's equation. By inserting a test particle in the mean flow and by calculating the averaged flow field around it we obtain a drag formula for the particles. This formula is found to depend only on the first three moments of the particle number density distribution function and thus can be computed readily for any given particle size distribution.

In § 6, the physical meaning of the mean flow equations is discussed in relation to Darcy's equation. We hope that this short discussion will help to provide a view of Darcy's empirical equation from a different standpoint and thus lead to a better understanding of fluid flow in porous media.

2. Statistical consideration

Let us now consider the question of the drag experienced by a cloud of N particles in low Reynolds number flow. If N is a small number then the problem can be handled by a straightforward extension of the technique outlined in the appendix. However, if N is large this technique of summing up directly the disturbances produced by all particles except the one in which we are interested becomes not only difficult but is also meaningless. When we have a cloud of particles it is, first of all, an almost impossible task to assign a definite position to each particle. Further, if the size of the particles are not the same it is also very unlikely that we can specify the size of a certain particle located at a certain position in space. The best that we can do is to characterize the cloud of particles by its statistical properties. For instance, if σ denotes the radius of sphere, then we can describe the size distribution of the cloud of spherical particles by a size distribution function $n(\mathbf{x}, \sigma)$ such that $n(\mathbf{x}, \sigma)\, d\sigma$ is the fraction of the number of particles with radius in the range σ to $\sigma + d\sigma$ in the neighbourhood of $\mathbf{x}$. Equivalently, the probability of finding a particle in the neighbourhood of $\mathbf{x}$ with radius σ to $d\sigma$ is proportional to $n(\mathbf{x}, \sigma)\, d\sigma$. As is well known from statistical mechanics, if the number of particles is sufficiently large the many-body problem can become greatly simplified since statistical concepts can then be applied. In the present problem the same situation arises. But before we go any further it is necessary to define clearly certain statistical quantities which we will use in the next few sections.

Let us denote the positions and radii of the N particles by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_N$, $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_N$. Here we will consider the parameter σ to be continuous. The probability that the set of N particles will be located in the volume element $d\mathbf{x}_1, d\mathbf{x}_2, d\mathbf{x}_3, \ldots, d\mathbf{x}_N$ with radii in the region $\sigma_1, \sigma_2, \ldots, \sigma_N$ is given by

$$p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N; \sigma_1, \sigma_2, \ldots, \sigma_N)\, d\mathbf{x}_1\, d\mathbf{x}_2, \ldots, d\mathbf{x}_N, d\sigma_1, d\sigma_2, \ldots, d\sigma_N.$$

We will refer to the space formed by $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_N, \sigma_1, \sigma_2, \ldots, \sigma_N$ as configuration space. The probability distribution of a single particle may be obtained by integrating over all other particles in the configuration space, e.g.

$$p(\mathbf{x}_1, \sigma_1) = \int p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N; \sigma_1, \sigma_2, \ldots, \sigma_N)\, d\mathbf{x}_2, d\mathbf{x}_3, \ldots, d\mathbf{x}_N, d\sigma_2, d\sigma_3, \ldots, d\sigma_N.$$

In what follows we will make the assumption that the distribution of particles is random, that is:

$$p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N; \sigma_1, \sigma_2, \ldots, \sigma_N) = p(\mathbf{x}_1, \sigma_1)\, p(\mathbf{x}_2, \sigma_2) \ldots p(\mathbf{x}_N, \sigma_N).$$

This assumption cannot be rigorously correct for spherical particles of finite radius, i.e. it is correct only when the particles are points that occupy no space at all. However, as we will soon use a point force approximation (see appendix) by assuming that the disturbance produced by a particle can be approximated by that produced by a point force, we feel that the two assumptions are at least compatible with each other and there is no reason to make things more complicated at this stage. (We expect this assumption to break down when the volume concentration of particles is too large.)

The ensemble average of a quantity ϕ will be defined as

$$\langle \phi(\mathbf{x}) \rangle = \int p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N; \sigma_1, \sigma_2, \ldots, \sigma_N)\, \phi(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N; \sigma_1, \ldots, \sigma_N)$$
$$d\mathbf{x}_1, d\mathbf{x}_2, \ldots, d\mathbf{x}_N, d\sigma_1, d\sigma_2, \ldots, d\sigma_N.$$

Also

$$\langle \phi^{(1)}(x) \rangle\, p(x_1, \sigma_1) = \int p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N; \sigma_1, \sigma_2, \ldots, \sigma_N)\, \phi\, d\mathbf{x}_2, d\mathbf{x}_3, \ldots, d\mathbf{x}_N,$$
$$d\sigma_2, \ldots, d\sigma_N.$$

It is to be noted that probability distribution under the above assumption may be related to density distribution by

$$n(\mathbf{x}, \sigma) = Np(\mathbf{x}, \sigma),$$

where $n(\mathbf{x}, \sigma)\, d\sigma$ is the fraction of number density of particles with radii between σ and $\sigma + d\sigma$ in the neighbourhood of $\mathbf{x}$.

3. Fundamental equations

We will now list explicitly the two important assumptions we want to make before we proceed to derive the fundamental equations of particle interaction in slow viscous flow. (i) The disturbance produced by a particle is to be approximated by that of a point force located at the centre of the spherical particle. The force is taken to be equal in magnitude but opposite in direction to the drag on the particle. (ii) The drag experienced by a particle is approximately directly proportional to the unperturbed velocity of the fluid (as seen by the particle) at the centre of the particle. (This includes the fluid velocity in the absence of all particles and the perturbed fluid velocities due to all other particles.)

The same approximations are used in the appendix. By assumption (ii), we can write

$$\text{Drag} = D(\sigma)\, U_i, \tag{1}$$

(we will use subscript index i and j to denote a vector in this section) where U_i is the unperturbed fluid velocity at the centre of the particle (in the sense of assumption (ii)) and $D(\sigma)$ is a coefficient which depends on the size of the particle and the statistical properties of the N particles. At this stage the dependence of D on σ is still unknown and in fact it is our aim to find it.

Let $U_{ij}(\mathbf{x}, \mathbf{x}_n)$ be the velocity of fluid in the ith direction at $\mathbf{x}$, due to a point force of unit strength in the jth direction at the point $\mathbf{x}_n$. The exact expression for $U_{ij}(\mathbf{x}, \mathbf{x}_n)$† can be obtained from (A 4). Also denote by $P_j(\mathbf{x}, \mathbf{x}_n)$ the corresponding pressure, so that U_{ij}, P_j satisfy Stokes' equation in the following form

$$\mu \frac{\partial^2}{\partial x_k^2}(U_{ij}(\mathbf{x}, \mathbf{x}_n)) - \frac{\partial}{\partial x_i}(p_j(\mathbf{x}, \mathbf{x}_n)) = -\delta(\mathbf{x}-\mathbf{x}_n)\,\delta_{ij}, \tag{2}$$

$$\frac{\partial}{\partial x_i}(U_{ij}(\mathbf{x}, \mathbf{x}_n)) = 0, \tag{3}$$

(repeated subscripts indicate summation).

By means of assumptions (i) and (ii) the fluid velocity and pressure at a point $\mathbf{x}$ is equal to

$$U_i(\mathbf{x}) = U_i^{(0)}(\mathbf{x}) - \sum_{n=1}^{N} G_j^{(n)}\, U_{ij}(\mathbf{x}, \mathbf{x}_n), \tag{4}$$

$$p(\mathbf{x}) = p^{(0)}(\mathbf{x}) - \sum_{n=1}^{N} G_j^{(n)}\, P_j(\mathbf{x}, \mathbf{x}_n), \tag{5}$$

where $U^{(0)}(\mathbf{x})$, $p^{(0)}(\mathbf{x})$ are the fluid velocity and pressure in the absence of the particles. $G_i^{(n)}$ is the drag on particle n in the ith direction,

$$G_i^{(n)} = D(\sigma_n)\, U_i^{(n)}(\mathbf{x}_n), \tag{6}$$

where

$$U_i^{(n)}(\mathbf{x}) = U_i^{(0)}(\mathbf{x}) - \sum_m{}' G_j^{(m)} U_{ij}(\mathbf{x}, \mathbf{x}_m). \tag{7}$$

($\sum'$ means that the term $m = n$ is to be omitted from the summation.) In other words, $U_i^{(n)}(\mathbf{x}_n)$ is the unperturbed fluid velocity as seen by particle n.

It is to be noted the $U_i^{(n)}(\mathbf{x})$ is not singular at $\mathbf{x} = \mathbf{x}_n$ even though $U_i(\mathbf{x})$ is.

By substituting (6) into (4) and (5) we obtain the fundamental equations of particle interaction in low Reynolds number flow.

$$U_i(\mathbf{x}) = U_i^{(0)}(\mathbf{x}) - \sum_n D_n\, U_j^{(n)}(\mathbf{x}_n)\, U_{ij}(\mathbf{x}, \mathbf{x}_n), \tag{8}$$

$$p(\mathbf{x}) = p^{(0)}(\mathbf{x}) - \sum_n D_n U_j^{(n)}(\mathbf{x}_n)\, P_j(\mathbf{x}, \mathbf{x}_n), \tag{9}$$

$$U_i^{(n)}(\mathbf{x}) = U_i^{(0)}(\mathbf{x}) - \sum_m{}' D_m\, U_j^{(m)}(\mathbf{x}_m)\, U_{ij}(\mathbf{x}, \mathbf{x}_m). \tag{10}$$

This is a set of self-consistent algebraic equations through which the fluid velocity and pressure as well as the drag on each particle can in principle be found if a way is specified (e.g. as in the appendix) to relate $U_i^{(n)}(\mathbf{x})$ to the drag on particle n.

4. Averaged fluid equations

Equations (8), (9) and (10) as they stand do not in any way help us towards solving the present problem. The direct method of the appendix would be at this stage, to solve the set of equations contained in (10) and then determine the

† Note that $U_{ij}(\mathbf{x}, \mathbf{x}_n)$ is a function only of $\mathbf{x}-\mathbf{x}_n$.

fluid velocity and pressure from (8) and (9). However, here we will adopt an alternative method which has been used extensively in multiple wave scattering problems (see Foldy 1945; Lax 1951; Twersky 1964). In this method we take the ensemble average of (8) and (9) over the whole configuration space to obtain formally

$$\langle U_i(\mathbf{x})\rangle = U_i^{(0)}(\mathbf{x}) - \sum_n \iint D_n \langle U_j^{(n)}(\mathbf{x}_n)\rangle\, U_{ij}(\mathbf{x},\mathbf{x}_n)\frac{n(\mathbf{x}_n,\sigma_n)}{N}\,d\mathbf{x}_n\,d\sigma_n, \tag{11}$$

$$\langle p(\mathbf{x})\rangle = p^{(0)}(\mathbf{x}) - \sum_n \iint D_n \langle U_j^{(n)}(\mathbf{x}_n)\rangle\, p_j(\mathbf{x},\mathbf{x}_n)\frac{n(\mathbf{x}_n,\sigma_n)}{N}\,d\mathbf{x}_n\,d\sigma_n. \tag{12}$$

The quantity $\langle U_i^{(n)}(\mathbf{x}_n)\rangle$ on the right-hand sides of (11) and (12) is the averaged fluid velocity as seen by particle n. It differs from the averaged fluid velocity $\langle U_i(\mathbf{x}_n)\rangle$† by a term of order $1/N$. If N is a large number then a good approximation is to substitute $\langle U_i(\mathbf{x}_n)\rangle$ for $\langle U_i^{(n)}(\mathbf{x}_n)\rangle$ in the right-hand sides of (11) and (12). Here we will assume that this is valid although we cannot prove it to be so. On making this approximation we obtain two integral equations for $\langle U_i(\mathbf{x})\rangle$ and $\langle p(\mathbf{x})\rangle$

$$\langle U_i(\mathbf{x})\rangle = U_i^{(0)}(\mathbf{x}) - \int F(\mathbf{x}_n)\langle U_j(\mathbf{x}_n)\rangle\, U_{ij}(\mathbf{x},\mathbf{x}_n)\,d\mathbf{x}_n, \tag{13}$$

$$\langle p(\mathbf{x})\rangle = p^{(0)}(\mathbf{x}) - \int F(\mathbf{x}_n)\langle U_j(\mathbf{x}_n)\rangle\, P_j(\mathbf{x},\mathbf{x}_n)\,d\mathbf{x}_n, \tag{14}$$

where

$$F(\mathbf{x}) = \int n(\mathbf{x},\sigma)\,D(\sigma)\,d\sigma. \tag{15}$$

Let us now apply the operators of Laplacian, gradient and divergence to (13) and (14). With the help of (2) and (3), a straightforward computation shows that $\langle U_i(\mathbf{x})\rangle$ and $\langle p(\mathbf{x})\rangle$ satisfy the following differential equations.

$$\mu\nabla^2\langle \mathbf{U}(\mathbf{x})\rangle - \nabla\langle p(\mathbf{x})\rangle = F(\mathbf{x})\langle \mathbf{U}(\mathbf{x})\rangle, \tag{16}$$

$$\nabla\,.\langle \mathbf{U}(\mathbf{x})\rangle = 0. \tag{17}$$

Equation (16) is in the form of a generalized Darcy's equation which describes the flow of fluid in a porous medium. It is, of course, no great surprise that the mean fluid equations turn out to be in this form. After all, a particle cloud is a form of porous medium.

5. The drag formula

In this section we want to make use of the mean flow equations (16) and (17) to obtain a drag formula for a spherical particle in a homogeneous isotropic particle cloud. For this purpose let us imagine a test particle of radius σ to be added to the cloud of particles. Also let the mean fluid velocity before the addition of the test particle be equal to $U_0\,\hat{e}_x$. Then outside the spherical particle the mean fluid flow is described by (16) and (17) which can be written as

$$\mu(\nabla^2\langle \mathbf{U}\rangle - \alpha^2\langle \mathbf{U}\rangle) = \nabla\langle p\rangle, \tag{18}$$

$$\nabla\,.\langle \mathbf{U}\rangle = 0, \tag{19}$$

where

$$\mu\alpha^2 = \int n(\sigma)\,D(\sigma)\,d\sigma, \tag{20}$$

$n(\sigma)$ is the number density distribution of the particle cloud.

† Note $\langle U_i(\mathbf{x})\rangle$ is non-singular at $\mathbf{x}_n$ while $U_i(\mathbf{x})$ is.

By integrating (1) over the configuration space of all particles except the one the drag of which we wish to know, we obtain (for the test particle)

$$\text{Drag} = D(\sigma)\, U_0\, \hat{e}_x. \tag{21}$$

The boundary conditions on $\langle \mathbf{U} \rangle$ are: (a) $\langle \mathbf{U} \rangle \to U_0 \hat{e}_x$ away from the test particle; (b) $\langle \mathbf{U} \rangle \to 0$ on the surface of test particle.

A solution of the aforementioned equations and boundary conditions is

$$\langle \mathbf{U} \rangle = \operatorname{grad}\left[\left(U_0 + A\,\frac{e^{-\alpha r}(1+\alpha r)-1}{\alpha^2 r^3} - \frac{B}{r^3}\right)x\right] + A\frac{e^{-\alpha r}}{r}\,\hat{e}_x, \tag{22}†$$

$$\langle p \rangle = \mu\left(\frac{A+\alpha^2 B}{r^3} - \alpha^2 U_0\right)x, \tag{23}$$

where

$$A = -\tfrac{3}{2}\sigma U_0\, e^{\alpha\sigma},$$

$$B = \frac{U_0 \sigma}{2\alpha^2}\left(-\alpha^2\sigma^2 + 3(e^{\alpha\sigma} - 1 - \alpha\sigma)\right).$$

By means of (22) and (23) the drag on the test sphere can be calculated which together with (21) gives

$$D(\sigma) = 6\pi\mu\sigma(1 + \alpha\sigma + \tfrac{1}{3}(\alpha^2\sigma^2)). \tag{24}$$

Now we can multiply (24) by $n(\sigma)$ and integrate over σ and solve for α. We obtain

$$\alpha = \frac{\{6\pi m_2 + [36\pi^2 m_2^2 + 24\pi m_1(1 - 3c/2)]^{\frac{1}{2}}\}}{(2-3c)}, \tag{25}$$

where m_1, m_2, m_3 are the first three moments of the distribution function $n(\sigma)$,

$$m_n = \int n(\sigma)\,\sigma^n\, d\sigma, \tag{26}$$

$$c = \tfrac{4}{3}\pi m_3 = \text{volume concentration of particles.}$$

For a given distribution of particles $n(\sigma)$ the drag on a particle of radius σ can now be calculated from (25), (24) and (21).

In the special case where all the particles are of the same size, say of radius a, the drag formula becomes

$$\begin{aligned}\text{Drag} &= 6\pi\mu a\left[\frac{4 + 3c + 3\sqrt{(8c - 3c^2)}}{(2-3c)^2}\right]\langle \mathbf{U}_0 \rangle \\ &= 6\pi\mu a \langle \mathbf{U}_0 \rangle \lambda''. \end{aligned} \tag{27}$$

Equation (27) was derived by Brinkman (1947) from a different starting point. Figure 1 shows a comparison of (27), Brinkman's formula, with experimental values given by Happel & Epstein (1954). Over the range of values of c where the basic assumptions used in obtaining (27) seem to be valid the agreement with experimental data is good. Equation (27) diverges as $c \to \frac{2}{3}$. At this value of c the particles are packed very close to each other. This, as expected, indicates the breakdown of the original assumptions of the point-force approximation and no particle–particle correlation. Except for this limitation the present derivation not only serves to provide a generalization of Brinkman's result but also tends to strengthen its theoretical value.

† We interpret this as the most probable flow field around the test particle.

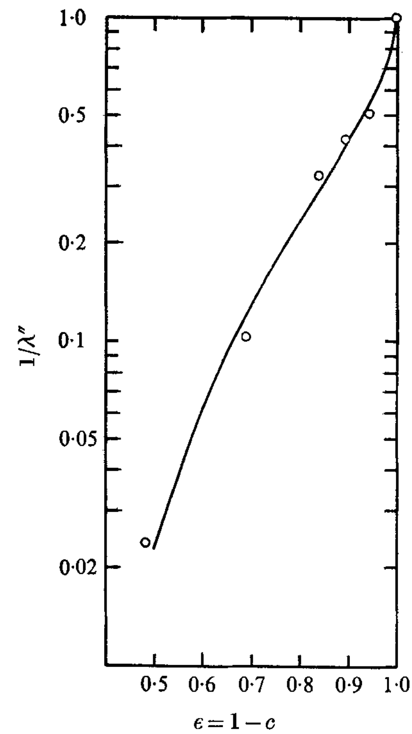

FIGURE 1. ○, experimental data, Happel & Epstein (1954); ——, equation (27), also Brinkman (1947).

6. Discussion

In this section we will try to understand the physical meaning of the mean fluid equations (16) and (17). The concept of permeability as introduced by Darcy does not seem to help us at all. In fact we tend to believe that if we try to interpret the effect of the linear term on the right-hand side of (16) by this concept we would probably have obscured its true significance. We feel that the real physical meaning of (16) and (17) can best be found by examining their fundamental solution. If a point force of magnitude D is applied at the origin inside a homogeneous isotropic cloud of particles then the averaged fluid velocity according to (16) and (17) is given by

$$\langle \mathbf{U}(\mathbf{x})\rangle = \frac{D}{4\pi\mu}\left(\frac{e^{-\alpha r}}{r}\hat{e}_x + \operatorname{grad}\frac{\partial}{\partial x}\frac{(1-e^{-\alpha r})}{\alpha^2 r}\right), \tag{28}$$

$$F(x) = \text{constant} = \mu\alpha^2, \quad r = \text{distance from origin.}$$

It is clear from (28) (cf. (A 4)) that the effect of the linear term on the right-hand side of (16) is to screen out any disturbances that happen to be produced inside the fluid. In fact a closer examination will show that the transverse velocity components† are effectively confined within a distance $1/\alpha$ (screening distance). For distances larger than $1/\alpha$ only the longitudinal velocity components matter.

† If $\mathbf{U} = \nabla\phi + \nabla\times\mathbf{A}$, transverse components refer to the second term.

Thus if the length scale of interest is larger than the screening distance we may neglect the transverse velocity components altogether. That is to say we can approximate (16) and (17) by

$$-\nabla\langle p\rangle = \mu\alpha^2\langle \mathbf{U}\rangle,$$
$$\nabla.\langle \mathbf{U}\rangle = 0.$$

(It can easily be shown that the term $\mu\nabla^2\langle \mathbf{U}\rangle$ does not contribute to the longitudinal velocity components.) We note that these are just Darcy's original equations for the flow in a porous medium. Here we have 'rediscovered' them through Stokes' equations and several other physical arguments. Though Darcy's equation is well known we hope that by deriving it in this fashion a deeper insight into the meaning of the equation has been gained.

The author wishes to thank Professors J. E. McCune and M. A. Hoffman for some valuable comments on this paper. This work was supported by the Ford Foundation.

Appendix

In this appendix we want to examine the point force approximation in a quantitative manner. We will compare results obtained by using this method with that given by known exact solution of Stokes' equation. Our purpose is not only to outline the present technique but also to show the rather astonishing accuracy such a simple approximation can give.

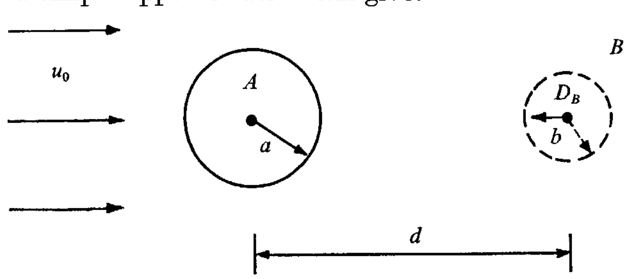

FIGURE 2. Flow past two spheres.

Stimson & Jeffery (1926) found an exact solution to the problem of Stokes flow past two spheres. The centres of the spheres are aligned in the direction of the flow. Let us examine this problem within the context of the point force approximation. We will denote the two spheres by A and B and their drag by D_A and D_B respectively. In considering the drag on sphere A we will first replace sphere B by a point force as in figure 2. Now to find D_A we need to solve the following boundary-value problem:

$$\mu\nabla^2\mathbf{U} - \nabla p = D_B\,\delta(x-d)\,\delta(y)\,\delta(z)\,\hat{e}_x, \tag{A 1}$$
$$\nabla.\mathbf{U} = 0, \tag{A 2}$$

where $\mathbf{U}$, p and μ are the fluid velocity, pressure and viscosity respectively. The boundary conditions are the usual no slip conditions on the surface of sphere A and $U \to U_0\,\hat{e}_x$ away from the sphere.

The solution of the above problem is rather simple if we make use of the fundamental solution of Stokes' equation corresponding to a point force in the x direction at $\mathbf{x}'$, namely

$$p = \frac{D}{4\pi}\frac{(x-x')}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{A 3}$$

$$\mathbf{U} = \frac{D}{8\pi\mu}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\hat{e}_x + \frac{(x-x')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right), \tag{A 4}$$

where D = magnitude of the force. All we need is to find a homogeneous solution to Stokes' equations which tends to zero away from the sphere and which when added on to (A 4) and the uniform flow $U_0\hat{e}_x$ will satisfy the no-slip condition on the surface of the sphere. Such a solution is available in Lamb (1945). Lamb's general solution is an expansion in orthogonal spherical harmonics which we can make full use of in the following way. If we expand the unperturbed velocity, as seen by sphere A, on its surface in terms of spherical harmonics and match it with Lamb's solution so that the no-slip condition is fulfilled we can readily show from this solution that the drag on the sphere is given by

$$\text{Drag} = 6\pi\mu a\mathbf{U}\text{ (unperturbed) mean over surface of sphere,} \tag{A 5}$$

where

$$\mathbf{U}_{\text{mean}}\text{ (unperturbed)} = \frac{1}{4\pi}\int_{\text{all solid angles}} \mathbf{U}(a,\theta,\phi)\text{ unperturbed } d\Omega.$$

By using (A 3), (A 4) and (A 5) and after straightforward integration we have

$$D_A = 6\pi\mu aU_0 + \frac{3}{2}\left(\frac{a^3}{3d^3}-\frac{a}{d}\right)D_B. \tag{A 6}$$

Similarly for sphere B we have

$$D_B = 6\pi\mu bU_0 + \frac{3}{2}\left(\frac{b^3}{3d^3}-\frac{b}{d}\right)D_A. \tag{A 7}$$

We can solve (A 6) and (A 7) simultaneously for D_A and D_B. If $a = b$ we have

$$D_A = D_B = \frac{6\pi\mu aU_0}{1+\frac{3}{2}([a/d]-[a^3/3d^3])} = 6\pi\mu aU_0\lambda. \tag{A 8}$$

We note from (A 5) that a further approximation can be made if we take

$$\begin{aligned}\mathbf{U}_{\text{mean}}\text{ (unperturbed)} &= \mathbf{U}\text{(unperturbed) at centre of sphere,}\\ &= U_0\hat{e}_x + \text{disturbance from other sphere.}\end{aligned}$$

On proceeding as before, instead of (A 7) we will obtain

$$D_A = D_B = \frac{6\pi\mu aU_0}{1+\frac{3}{2}(a/d)} = 6\pi\mu aU_0\lambda'. \tag{A 9}$$

Table 1 shows a comparison of the values given by (A 8) and (A 9) and that of the exact values of Stimson & Jeffery. It can be seen that the numerical agreement is good for almost all distance of separation of the two spheres. With this example in mind we feel a bit confident that this point force approximation would give satisfactory results for our present intended purpose.

$d/2a$	λ_{exact}	λ	λ'
1	0·645	0·593	0·571
1·543	0·702	0·681	0·673
2·352	0·768	0·761	0·758
3·762	0·836	0·835	0·834
6·132	0·892	0·891	0·891
10·068	0·931	0·930	0·930
∞	1	1	1

TABLE 1. (λ_{exact} is from Stimson & Jeffrey (1926).)

REFERENCES

BRINKMAN, H. C. 1947 A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Appl. Scientific Research*, **A1**, 27.

FOLDY, F. L. 1945 The multiple scattering of waves. *Phys. Rev.* **67**, 107.

HAPPEL, J. & BRENNER, H. 1965 *Low Reynolds Number Hydrodynamics*. New York: Prentice Hall.

HAPPEL, J. & EPSTEIN, N. 1954 Cubical assemblages of uniform spheres. *Ind. and Engng Chem.* **46**, 1187.

KELLER, J. B. 1964 Stochastic equations and wave propagation in random media. *Proc. of Symposia in Appl. Math.* vol. XVI. Amer. Math. Soc.

LAMB, H. 1945 *Hydrodynamics*. 6th ed. New York: Dover.

LAX, M. 1951 Multiple scattering of waves. *Rev. Modern Phys.* **23**, 287.

STIMSON, M. & JEFFERY, G. B. 1926 The motion of two spheres in a viscous fluid. *Proc. Roy. Soc. Lond.* **111**, 110.

TWERKSY, V. 1964 On propagation in random media of discrete scatters. *Proc. of Symposia in Appl. Math.* vol. XVI. Amer. Math. Soc.